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LETTER TO THE EDITOR

Identities involving elementary symmetric functions

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Abstract. A systematic procedure for generating certain identities involving elementary symmetric functions is proposed. These identities, as particular cases, lead to a hierarchy of identities for q-binomial coefficients.

Ever since the advent of Calogero–Sutherland models [1–4] there has been a considerable interest in finding homogeneous symmetric polynomials $P_k(x)$; $x \equiv (x_1, x_2, \ldots, x_N)$ of degree k which satisfy the generalized Laplace equation

$$\left[\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{i<j} \frac{1}{(x_i - x_j)} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)\right] P_k(x) = 0. \tag{1}$$

Since one is seeking solutions to (1) which are symmetric functions of $(x_1, x_2, \ldots, x_N)$ it appears natural to change variables from $(x_1, x_2, \ldots)$ to a set of variables which are symmetric functions of $(x_1, x_2, \ldots)$ and rewrite the generalized Laplace equation in terms of these variables. Two sets of such variables that have been considered in the literature [5, 6] are

- power sums:

$$p_r(x) = \sum_i x_i^r \qquad r = 1, \ldots, N \tag{2}$$

- elementary symmetric functions:

$$e_r(x) = \sum_{i_1 < i_2 \cdots < i_r} x_{i_1} x_{i_2} \ldots x_{i_r} \qquad i_1, \ldots, i_r = 1, \ldots, N \qquad r = 1, \ldots, N. \tag{3}$$

(Here, for symmetric functions, we follow the nomenclature and notation of [7].) Explicit expressions for the generalized Laplace equation in terms of these variables may be found in [5] and [6] respectively. The next step consists in finding polynomial solutions of the equation thus obtained. (It may be noted here that a more efficient way of constructing the symmetric polynomial solutions of (1) based on expanding $P_k(x)$ in terms of Jack polynomials [8] may be found in [9].)

In changing variables from $(x_1, \ldots, x_N)$ to $(e_1(x), \ldots, e_N(x))$ in the generalized Laplace equation, in the intermediate stages, one needs to express the symmetric function

$$\sum_i e^{(i)}_{p-1}(x) e^{(i)}_{q-1}(x) \tag{4}$$

in terms of $e_r(x)$. Here $e_p^{(i)}(x)$ denotes the pth elementary symmetric function formed from $(x_1, \ldots, x_N)$ omitting x_i. The purpose of this letter is to provide a derivation of the expression of the symmetric function in (4) in terms of the elementary symmetric functions in the full set of variables $(x_1, \ldots, x_N)$. The procedure adopted for deriving this result permits easy extension to symmetric functions such as

$$\sum_{i=1}^{N} e_{p-1}^{(i)}(x) e_{q-1}^{(i)}(x) e_{r-1}^{(i)}(x) \tag{5}$$

and so on. Further, on setting $x_1 = 1, x_2 = q, \ldots, x_N = q^{N-1}$, one is led to a series of interesting identities for q-binomial coefficients.

To obtain the desired results, it proves convenient to work with the generating function for the elementary symmetric functions

$$E(x,t) = \sum_{r=0}^{N} t^r e_r(x) \tag{6}$$

$$= \prod_{i=1}^{N} (1 + x_i t). \tag{7}$$

From the product structure of $E(x,t)$ it follows that

$$e_p(x_1, x_2, \ldots, x_N) = \sum_{l=0}^{p} e_l(x_1, x_2, \ldots, x_i) e_{p-l}(x_{i+1}, \ldots, x_N). \tag{8}$$

Differentiating $\log E(x,t)$ with respect to t gives

$$\frac{\partial}{\partial t} \log E(x,t) = \sum_{i=1}^{N} \frac{x_i}{(1 + x_i t)}. \tag{9}$$

Further, differentiating $\log E(x,t)$ with respect to x_i one obtains

$$\frac{\partial}{\partial x_i} \log E(x,t) = \frac{t}{(1 + x_i t)} \tag{10}$$

and hence

$$\prod_{\alpha=1}^{M} \left[\frac{\partial}{\partial x_i} \log E(x, t_\alpha) \right] = \left(\prod_{\alpha=1}^{M} t_\alpha \right) \left(\prod_{\alpha=1}^{M} \frac{1}{1 + x_i t_\alpha} \right). \tag{11}$$

Our aim now is to express the rhs of (11) in terms of derivatives of $\log E(x,t)$ with respect to t. To this end, we notice that the second product on the rhs of (11) can be expressed as follows:

$$\left(\prod_{\alpha=1}^{M} \frac{1}{1 + x_i t_\alpha} \right) = 1 + \sum_{\alpha}^{M} f_\alpha(t) \frac{x_i}{1 + x_i t_\alpha} \tag{12}$$

where $f_\alpha(t)$ satisfy the following set of linear equations:

$$\begin{aligned}
&\sum_\alpha f_\alpha = -e_1(t) \\
&\sum_\alpha f_\alpha e_1^{(\alpha)}(t) = -e_2(t) \\
&\sum_\alpha f_\alpha e_2^{(\alpha)}(t) = -e_3(t) \\
&\vdots \\
&\sum_\alpha f_\alpha e_{M-1}^{(\alpha)} = -e_M(t).
\end{aligned} \tag{13}$$

The solution of this set of linear equations turns out to be remarkably simple:

$$f_\alpha(t) = (-t_\alpha)^M \prod_{\beta\neq\alpha} \frac{1}{(t_\beta - t_\alpha)}. \tag{14}$$

Using (12) in (11), summing over i, and using (9) we obtain

$$\sum_i \prod_{\alpha=1}^{M} \left[\frac{\partial}{\partial x_i} \log E(x, t_\alpha)\right] = \left(\prod_{\alpha=1}^{M} t_\alpha\right)\left[N + \sum_{\alpha=1}^{M} f_\alpha \frac{\partial}{\partial t_\alpha} \log E(x, t_\alpha)\right] \tag{15}$$

i.e.

$$\sum_i \prod_{\alpha=1}^{M} \left[\frac{\partial}{\partial x_i} E(x, t_\alpha)\right] = N \prod_{\alpha=1}^{M} t_\alpha E(x, t_\alpha) + \left(\prod_{\alpha=1}^{M} t_\alpha\right) \sum_{\alpha=1}^{M} f_\alpha \frac{\partial}{\partial t_\alpha} \prod_{\beta=1}^{M} E(x, t_\beta) \tag{16}$$

where the f are given by (14). This relation is a rich source of a hierarchy of identities involving elementary symmetric functions and hence that for q-binomial and binomial coefficients as can be seen from the following illustrative examples.

Consider first the simplest of the hierarchy of identities implied by (16) obtained by setting $\alpha = 1$. In this case, (16) yields

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} E(x, t) = N t E(x, t) - t^2 \frac{\partial}{\partial t} E(x, t). \tag{17}$$

On substituting for $E(x, t)$ from (6) and equating like powers of t on both sides one obtains

$$\sum_{i=1}^{N} e_{p-1}^{(i)}(x) = (N - p + 1) e_{p-1}(x). \tag{18}$$

Now, from (8) it follows that

$$e_{p-1}^{(i)}(x) = \sum_{l=1}^{p} e_{l-1}(x_1, x_2, \ldots, x_{i-1}) e_{p-l}(x_{i+1}, \ldots, x_N) \tag{19}$$

and hence

$$\sum_{i=1}^{N} e_{p-1}^{(i)}(x) = \sum_{l=1}^{p} \sum_{i=1}^{N} e_{l-1}(x_1, x_2, \ldots, x_{i-1}) e_{p-l}(x_{i+1}, \ldots, x_N). \tag{20}$$

Using the fact that $e_l(x_1, \ldots, x_i)$ is nonzero only if $i \geqslant l$, we can rewrite (20), after some rearrangement, as

$$\sum_{i=1}^{N} e_{p-1}^{(i)}(x) = \sum_{i=1}^{N-p+1} \sum_{l=0}^{p-1} e_l(x_1, x_2, \ldots, x_{i+l-1}) e_{p-l-1}(x_{i+l+1}, \ldots, x_N). \tag{21}$$

On using this result in (18) we obtain

$$\sum_{i=1}^{N-p+1} \sum_{l=0}^{p-1} e_l(x_1, x_2, \ldots, x_{i+l-1}) e_{p-l-1}(x_{i+l+1}, \ldots, x_N) = (N - p + 1) e_{p-1}(x). \tag{22}$$

Setting $x_1 = 1, x_2 = q, \ldots, x_N = q^{N-1}$ and using

$$e_p(1, q, \ldots, q^{N-1}) = q^{p(p-1)/2} \begin{bmatrix} N \\ p \end{bmatrix} \tag{23}$$

where

$$\begin{bmatrix} N \\ p \end{bmatrix} \equiv \frac{(1 - q^N)(1 - q^{N-1}) \cdots (1 - q^{N-p+1})}{(1 - q)(1 - q^2) \cdots (1 - q^p)} \tag{24}$$

denotes the q-binomial coefficient [7, 10], we obtain, on changing $p-1$ to p, the following q-binomial identity:

$$\sum_{i=1}^{N-p}\sum_{l=0}^{p} q^{il}\begin{bmatrix} N-p-i+l \\ l \end{bmatrix}\begin{bmatrix} i-1+p-l \\ p-l \end{bmatrix} = (N-p)\begin{bmatrix} N \\ p \end{bmatrix}. \tag{25}$$

This identity has a totally different structure as compared with that obtained from (8). For $N-p \geqslant i \geqslant 1$ (8) yields

$$\sum_{l=0}^{i} q^{(i-l)(p-l)}\begin{bmatrix} N-i \\ p-l \end{bmatrix}\begin{bmatrix} i \\ l \end{bmatrix} = \begin{bmatrix} N \\ p \end{bmatrix} \tag{26}$$

which on summing over i from 1 to $N-p$ gives

$$\sum_{i=1}^{N-p}\sum_{l=0}^{i} q^{(i-l)(p-l)}\begin{bmatrix} N-i \\ p-l \end{bmatrix}\begin{bmatrix} i \\ l \end{bmatrix} = (N-p)\begin{bmatrix} N \\ p \end{bmatrix}. \tag{27}$$

Notice that (22) can be rewritten as

$$\sum_{i=1}^{N-p+1}\left[\sum_{l=0}^{p-1} e_l(x_1, x_2, \ldots, x_{i+l-1})e_{p-l-1}(x_{i+l+1}, \ldots, x_N) - e_{p-1}(x)\right] = 0 \tag{28}$$

suggesting the following identitity:

$$e_p(x) = \sum_{l=0}^{p} e_l(x_1, x_2, \ldots, x_{i+l-1})e_{p-l}(x_{i+l+1}, \ldots, x_N) \tag{29}$$

valid for $N-p \geqslant i \geqslant 1$. This gives rise to the following q-binomial identity:

$$\sum_{l=0}^{i} q^{(i-l)(p-l)}\begin{bmatrix} N-i \\ p-l \end{bmatrix}\begin{bmatrix} i \\ l \end{bmatrix} = \begin{bmatrix} N \\ p \end{bmatrix} \qquad N-p \geqslant i \geqslant 1. \tag{30}$$

From (18) we can derive more identities by differentiating with respect to x_j, summing over j and using (18) on the rhs of the relation thus obtained. Repeating this procedure one is led to

$$\sum_{i,j=1;i_1\cdots>i_r}^{N} e_{p-r}^{(i_1,\ldots,i_r)}(x) = \binom{N-p+r}{r} e_{p-r}(x). \tag{31}$$

Setting $x_1 = 1, x_2 = q, \ldots, x_N = q^{N-1}$, as before, one obtains a whole series of q-binomial identities.

The next in the hiearchy of identities corresponds to $\alpha = 2$. In this case (16) reads

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}E(x,t_1)\frac{\partial}{\partial x_i}E(x,t_2) = Nt_1t_2E(x,t_1)E(x,t_2) + t_1t_2\left[\frac{t_1^2}{t_2-t_1}\frac{\partial}{\partial t_1} + \frac{t_2^2}{t_1-t_2}\frac{\partial}{\partial t_2}\right] \times E(x,t_1)E(x,t_2). \tag{32}$$

On substituting from (6) and equating like powers of t_1 and t_2 on both sides one obtains

$$\sum_{i=1}^{N} e_{p-1}^{(i)}(x)e_{q-1}^{(i)}(x) = (N-p+1)e_{p-1}(x)e_{q-1}(x) - \sum_{l=0}^{q-2}(p+q-2-2l)e_{p+q-2-l}(x)e_l(x) \tag{33}$$

which is the desired result valid for $p \geqslant q \geqslant 2$.

For the case $\alpha = 3$, (16) gives

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} E(x,t_1) \frac{\partial}{\partial x_i} E(x,t_2) \frac{\partial}{\partial x_i} E(x,t_3) = N t_1 t_2 t_3 E(x,t_1) E(x,t_2) E(x,t_3) - t_1 t_2 t_3 \times \left[\frac{t_1^3}{(t_3 - t_1)(t_2 - t_1)} \frac{\partial}{\partial t_1} + \frac{t_2^3}{(t_3 - t_2)(t_1 - t_2)} \frac{\partial}{\partial t_2} + \frac{t_3^3}{(t_1 - t_3)(t_2 - t_3)} \frac{\partial}{\partial t_3} \right] E(x,t_1) E(x,t_2) E(x,t_3) \tag{34}$$

which, in turn, yields

$$\sum_{i=1}^{N} e_{p-1}^{(i)}(x) e_{q-1}^{(i)}(x) e_{r-1}^{(i)}(x) = \sum_{m=1}^{r-3} \sum_{l=1}^{m} l e_l e_{m+q-l} e_{p+r-m-3} - \sum_{m=0}^{r-1} \sum_{l=1}^{q-3} l e_l e_{m+p+q-l-2} e_{r-m-1} - \sum_{m=0}^{r-1} \sum_{l=0}^{m} (m+q-l-2) e_l e_{m+q-l-2} e_{p+r-m-1} + \sum_{m=0}^{r-1} \sum_{l=0}^{q-1} (m+p+q-l-2) e_l e_{m+p+q-l-2} e_{r-m-1} \tag{35}$$

valid for $p \geqslant q \geqslant r$. Again, as before, we can derive identities for q-binomial coefficients by setting $x_i = q^{i-1}$ in (33) and (35).

To conclude, we have developed a systematic procedure for expressing sums of products of elementary symmetric functions of the form

$$\sum_{i}^{N} e_{p-1}^{(i)}(x) e_{q-1}^{(i)}(x) \cdots e_{w-1}^{(i)}(x) \tag{36}$$

in terms of elementary symmetric functions in the full set of variables $x_1, \ldots, x_N$. All such relations are derivable from (16), which constitutes the central result of this letter. These relations, in turn, are shown to lead to a hierarchy of identities involving q-binomial coefficients.

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